Solving equations numerically

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This note describes methods for solving equations numerically, when there is no analytical solution, that might be encountered in Examples Papers or Tripos questions. Suppose we seek a solution for x satisfying,

$$f(x) = x^2 - 3 - \ln x = 0 \ . \tag{1}$$

WolframAlpha The quickest approach is to type a query (clickable link) into the WolframAlpha website. This shows a nice graph of the function and gives two solutions, $x = 0.0499$ or $x = 1.91$. WolframAlpha is also good for checking integrals, limits, or unwieldy algebra manipulations.

Trial and improvement In Tripos exams, we only have access to a calculator. The slowest but most robust way to find a solution is trial and improvement. We guess different values of x, evaluate the function, compare to zero, and refine our guesses until we reach a desired precision,

$f(x = 1)$	$= -2$	too small;
$f(x = 2)$	$= 0.307$	too big, but closer;
$f(x = 1.75)$	$= -0.497$	too small, but closer;
$f(x = 1.9)$	$= -0.032$	close enough.

Fixed-point iteration Rearranging to make an x the subject of Eqn. (1), $x_{i+1} = \sqrt{\ln x_i + 3}$, where we have indexed values of x over i to show that they form a sequence. We can calculate successive terms in this sequence quickly with our calculator by using the `Ans` variable to store x_i and repeatedly hitting the `=` button. The inputs are,

Key input	*Screen output*
`1 =`	1
`sqrt(ln(Ans)+3) =`	$\sqrt{3}$
`=`	1.91
`=`	1.91

Another rearrangement of Eqn. (1), $x_{i+1} = \exp\left(x^2 - 3\right)$, finds the second root.

Calculator solve Your calculator is equipped with a solve function, which is the easiest and fastest way to tackle the problem. The steps are,

- Enter the equation using `X` as the unknown to solve for with `ALPHA, )`, and a literal equals sign using the `ALPHA, CALC` keys;
- Activate the `SOLVE` function using `SHIFT, CALC` and you will be prompted for an initial guess, enter the guess and press normal equals;
- After a delay, the solution appears on the screen.

For our problem, the inputs are,

Key input	*Screen output*
`X`^{2} `- 3 - ln(X) ALPHA CALC 0 SHIFT CALC`	Solve for X
`1 =` (our initial guess)	1.91