# IB Paper 7: Vector Calculus
# Integrals How-to Guide

James Brind
jb753@cam.ac.uk

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**Abstract**

This document describes approaches for directly evaluating line and surface integrals, and provides two further examples. The steps in both cases are analogous. First, choose a parametrisation; a line needs one parameter, a surface two parameters. Next, determine the position vector and vector field as a function of the parameter(s). Take the derivative(s) of the position vector with respect to the parameter(s) to determine the vector length or area element. Finally, take the dot product and perform the integral.

## Line integrals

**Definition.** In three-dimensional space, given a vector field $\mathbf{f}$ and a line $L$, the 'line integral' of $\mathbf{f}$ along $L$ is given by,

$$\phi = \int_L \mathbf{f} \cdot \mathrm{d}\mathbf{L},$$

where $\mathrm{d}\mathbf{L}$ is an infinitesimal vector length element of the line $L$. We can write $\mathrm{d}\mathbf{L}$ as, $\mathrm{d}\mathbf{L} = \hat{\mathbf{e}}\mathrm{d}L$, where $\mathrm{d}L$ is the scalar arc length and $\hat{\mathbf{e}}$ is the unit tangential vector along $L$. Vector length is a pseudovector, with a magnitude equal to the scalar length of the element, and a direction tangent to the line at a particular point. Vector length defined in this way has the useful property that the *projected length* in any direction is the *component of the vector* in that direction. Since $\mathrm{d}\mathbf{L}$ is a pseudovector, there are two possible orientations, and hence two possible signs for the integral $\phi$. Which direction to choose is determined by convention or is evident from the details of the problem.

**Line.** A curve is a one-dimensional object, in that the distance along the curve is sufficient to identify any particular point on it (assuming we know where to start from). A *single parameter* is used to parametrise a curve. This could be distance, a convenient function of distance, or something else. What is required is that we must be able to associate each point on the curve to a unique value of the parameter, and the parameter should vary monotonically and continuously from the start to the end of the curve. Using a parameter $t$, we can specify all point on a curve as,

$$\mathbf{x}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k},$$

To illustrate, in Examples Paper 1, Question 9, a helix is defined,

$$\mathbf{x}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + kt\,\mathbf{k}, \quad 0 \le t \le 2\pi.$$

**Field.** To perform the integral with respect to the parameter, the vector field $\mathbf{f}$ as a function of $t$ needs to be found. Given the curve parametrisation, this is straightforward, as, with $\mathbf{f}(x,y,z)$, we can substitute in $\mathbf{f}(t) = \mathbf{f}(x(t),y(t),z(t))$. In Examples Paper 1, Question 9, the field is,

$$\mathbf{f}(x,y,z) = ky\,\mathbf{i} + kx\,\mathbf{j} + 2xy\,\mathbf{k},$$
$$\Rightarrow \mathbf{f}(t) = ka\sin t\,\mathbf{i} + ka\cos t\,\mathbf{j} + 2a^2\cos t\sin t\,\mathbf{k}.$$

**Vector line element.** Consider a small change in parameter $\delta t$. Use a first-order Taylor expansion to determine the resulting change in the position vector on the curve, and take the limit $\delta t \to 0$. There is more detail in the Lecture Notes. This yields the following intuitive result,

$$\mathrm{d}\mathbf{L} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t.$$

In our example, this evaluates to,

$$\mathrm{d}\mathbf{L} = (-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + k\,\mathbf{k})\,\mathrm{d}t.$$

**Performing the integral.** Now we have all the components of $\phi$, so we substitute them in, multiply out the dot product and perform the integral,

$$\phi = \int_L \mathbf{f}\cdot\mathrm{d}L$$
$$\phi = \int_{t=0}^{t=2\pi} \left(ka\sin t\,\mathbf{i} + ka\cos t\,\mathbf{j} + 2a^2\cos t\sin t\,\mathbf{k}\right)\cdot(-a\sin t\,\mathbf{i} + a\cos t\,\mathbf{j} + k\,\mathbf{k})\,\mathrm{d}t$$
$$\phi = \int_{t=0}^{t=2\pi} -ka^2\sin^2 t + ka^2\cos^2 t + 2a^2k\cos t\sin t\;\mathrm{d}t$$
$$\phi = ka^2\int_{t=0}^{t=2\pi} \sin 2t + \cos 2t\;\mathrm{d}t = \underline{\underline{0}}$$

**Summary.** The steps for performing a line integral of a vector field are,

1. Choose a parametrisation for the curve (if one is not given to you already). Express the position vector on the curve and vector field as a function of this parameter.
2. Determine the vector line element using the partial derivative of the position vector with respect to the parameter.
3. Substitute into the definition of line integral, multiply out the dot product, and perform the integration.

## Example line integral

*Evaluate the line integral of the vector field* $\mathbf{F} = 2xyz\,\mathbf{i} + x^2 z\,\mathbf{j} + x^2 y\,\mathbf{k}$ *over the straight line L joining the origin to the point* $\mathbf{a} = \mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$.

1. In the $x$–$y$ plane, straight lines through the origin are of the form $y = Cx$ where $C$ is a gradient constant. The equivalent result in three dimensions is $\mathbf{x} = \mathbf{c}x$, where $\mathbf{c}$ is a gradient vector. In this case, we choose our parameter $t = x$ and the position vector along the line is,

$$\mathbf{x}(t) = \mathbf{a}t = t\,\mathbf{i} - 2t\,\mathbf{j} - 3t\,\mathbf{k}, \quad 0 \le t \le 1.$$

   The corresponding expression for the vector field is,

$$\mathbf{F}(t) = 12t^3\,\mathbf{i} - 3t^3\,\mathbf{j} - 2t^3\,\mathbf{k}.$$

2. The vector length element is,

$$\mathrm{d}\mathbf{L} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}t}\left[t\,\mathbf{i} - 2t\,\mathbf{j} - 3t\,\mathbf{k}\right]\mathrm{d}t = (\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k})\,\mathrm{d}t.$$

3. Substituting the components in and performing the integral,

$$\phi = \int_L \mathbf{F}\cdot\mathrm{d}L = \int_{t=0}^{t=1} \left(12t^3\,\mathbf{i} - 3t^3\,\mathbf{j} - 2t^3\,\mathbf{k}\right)\cdot(\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k})\,\mathrm{d}t$$

$$\phi = \int_{t=0}^{t=1} 12t^3 + 6t^3 + 6t^3\ \mathrm{d}t = \int_{t=0}^{t=1} 24t^3\ \mathrm{d}t = \left[6t^4\right]_{t=0}^{t=1} = \underline{\underline{6}}$$

## Surface integrals

**Definition.** Given a vector field $\mathbf{f}$ and a surface $S$, the 'surface integral' of $\mathbf{f}$ over $S$ is given by,

$$\Phi = \iint_S \mathbf{f}\cdot\mathrm{d}\mathbf{S},$$

where $\mathrm{d}\mathbf{S}$ is an infinitesimal vector area element of $S$. Vector area is a pseudovector, with magnitude equal to the scalar surface area of the element and a direction given by the normal to the surface. So, the infinitesimal vector area element can be written as,

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\mathrm{d}S,$$

where $\mathrm{d}S$ is an infinitesimal area element small enough to be taken as flat, and $\hat{\mathbf{n}}$ is a unit vector normal to that area. As vector area is a pseudovector, there are two possible normal vectors, pointing in opposite directions. The orientation of $\hat{\mathbf{n}}$, and the sign of the integral $\Phi$ is determined by a sign convention: $\hat{\mathbf{n}}$ points outwards from a closed surface. The convenient thing about vector area defined in this way is that the *projected area* in any direction is given by the *component of the vector* in that direction.

**Surface.** A surface is two-dimensional. A surface can be described by defining a function for each coordinate in terms of *two parameters*, $u$ and $v$, say. In Cartesian coordinates,

$$\mathbf{x}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}.$$

Any point on the surface can be uniquely identified by values of $u$ and $v$. We can make any number of valid choices for what $u$ and $v$ actually are. However, it is sensible to reuse two of our coordinate directions. For example, the surface $x = y$ from Examples Paper 2 can be parametrised using $u = y$ and $v = z$ as follows,

$$x = y \quad \Rightarrow \quad \mathbf{x}(y,z) = y\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

**Field.** The vector field will be given to you as a function of one or more coordinates,

$$\mathbf{f} = \mathbf{f}(x,y,z)$$

If the surface $S$ is parametrised using a different set of coordinates, the vector field should be expressed as a function of those coordinates. As example, we will use the field from Examples Paper 2, Question 10:

$$\mathbf{f}(x,y,z) = x\,\mathbf{i} - y\,\mathbf{j} \quad \Rightarrow \quad \text{on } x = y,\ \mathbf{f}(y,z) = y\,\mathbf{i} - y\,\mathbf{j}$$

**Vector area element.** If we have the surface defined in terms of the general parameters $\mathbf{x}(u,v)$ then the vector area element also needs to be put in terms of the parameters,

$$\mathrm{d}\mathbf{S} = \hat{\mathbf{n}}\mathrm{d}S = \mathbf{n}(u,v)\mathrm{d}u\mathrm{d}v$$

Note that the normal $\mathbf{n}(u,v)$ is not a unit vector any more. This is because the scalar area $\mathrm{d}S \neq \mathrm{d}u\mathrm{d}v$ in general, and so the surface must be projected into the $u$–$v$ plane to perform the integration with respect to those variables. How should we determine $\mathbf{n}(u,v)$? Recall the geometric interpretation of the cross product. Given two vectors lying within the surface, the cross product will yield a vector normal to the surface, with a magnitude equal to area of the parallelogram defined by those two vectors. The partial derivatives of the surface position vector yield two suitable vectors,

$$\delta\mathbf{x}_u \approx \left.\frac{\partial\mathbf{x}}{\partial u}\right|_v \delta u,$$
$$\delta\mathbf{x}_v \approx \left.\frac{\partial\mathbf{x}}{\partial v}\right|_u \delta v,$$

which can then be crossed to find the vector area of the parallelogram,

$$\delta S \approx \pm \left.\frac{\partial\mathbf{x}}{\partial u}\right|_v \times \left.\frac{\partial\mathbf{x}}{\partial v}\right|_u \delta u\delta v.$$

Letting the size of the element tend to zero, we arrive at the result in the Maths Data Book,

$$\mathrm{d}\mathbf{S} = \pm \left.\frac{\partial\mathbf{x}}{\partial u}\right|_v \times \left.\frac{\partial\mathbf{x}}{\partial v}\right|_u \mathrm{d}u\mathrm{d}v,$$

where the sign must be chosen to give the conventional direction. This equation yields the vector area element $\mathrm{d}\mathbf{S}(u,v)$ at every point on the surface as a function of any chosen parameters $u$ and $v$, given a definition of the surface in terms of $\mathbf{x}(u,v)$. In our example, the partial derivatives are,

$$\mathbf{x}(y,z) = y\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad \Rightarrow \quad \left.\frac{\partial \mathbf{x}}{\partial y}\right|_z = \mathbf{i} + \mathbf{j}, \quad \left.\frac{\partial \mathbf{x}}{\partial z}\right|_y = \mathbf{k},$$

and so the vector area element is,

$$\mathrm{d}\mathbf{S} = \pm(\mathbf{i}+\mathbf{j}) \times \mathbf{k}\,\mathrm{d}y\mathrm{d}z = \pm(\mathbf{i}-\mathbf{j})\,\mathrm{d}y\mathrm{d}z = (\mathbf{i}-\mathbf{j})\,\mathrm{d}y\mathrm{d}z,$$

where the positive sign has been chosen to match the outward normal convention for the volume in Examples Paper 2, Question 10.

**Performing the integral.** We now know all the components of our definition of $\Phi$,

$$\Phi = \iint_S \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \int_{y=1}^{y=2} \int_{z=0}^{z=1} (x\mathbf{i} - y\mathbf{j}) \cdot (\mathbf{i} - \mathbf{j})\,\mathrm{d}y\mathrm{d}z,$$

where the limits are just the maximum and minimum extents of the surface in each parametric direction. Multiplying out the dot product, and evaluating the integral,

$$\Phi = \int_{y=1}^{y=2} \int_{z=0}^{z=1} (y\mathbf{i} - y\mathbf{j}) \cdot (\mathbf{i} - \mathbf{j})\,\mathrm{d}y\mathrm{d}z$$
$$\Phi = \int_{y=1}^{y=2} \int_{z=0}^{z=1} (y + (-1)(-1)y)\,\mathrm{d}y\mathrm{d}z$$
$$\Phi = \int_{y=1}^{y=2} \int_{z=0}^{z=1} 2y\,\mathrm{d}y\mathrm{d}z = \underline{\underline{3}}.$$

**Summary.** The steps for performing a surface integral are,

1. Parametrise the surface as a position vector which is a function of two coordinates, and find the vector field as a function of those parameters.
2. Determine the vector area element by crossing the two partial derivatives of the position vector, using the formula from the Maths Data Book.
3. Deduce the limits, multiply out the dot product and perform the integral.

# Example surface integral

*Evaluate the flux of the vector field* $\mathbf{v}(x,y,z) = x\mathbf{i} - (z-4)\,\mathbf{j}$ *through the surface* $S$ *defined by* $x^2+y^2+z = 4$*, for* $z > 0$.

1. The presence of $x^2+y^2$ and $z$ in the specification of $S$ suggests that a parametrisation in cylindrical polar coordinates might be useful. Put the specification of $S$ in terms of cylindrical polars,

   $$x^2+y^2 = r^2 \;\text{ and }\; x^2+y^2+z = 4 \Rightarrow z = 4 - r^2$$

   Then using the definition of cylindrical polar coordinates,

   $$\mathbf{x}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (4-r^2)\,\mathbf{k}$$

The vector field is, again from the definition of cylindrical polars,

$$\mathbf{v}(r,\theta,z) = r\cos\theta\ \mathbf{i} - (z-4)\ \mathbf{j}.$$

2. The partial derivatives are,

$$\left.\frac{\partial \mathbf{x}}{\partial r}\right|_{\theta} = \cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j} - 2r\ \mathbf{k},$$
$$\left.\frac{\partial \mathbf{x}}{\partial \theta}\right|_{r} = -r\sin\theta\ \mathbf{i} + r\cos\theta\ \mathbf{j},$$

so the vector area element is,

$$\begin{aligned}
\mathrm{d}S &= \pm\left.\frac{\partial \mathbf{x}}{\partial r}\right|_{\theta} \times \left.\frac{\partial \mathbf{x}}{\partial \theta}\right|_{r} \mathrm{d}r\mathrm{d}\theta \\
&= \pm(\cos\theta\ \mathbf{i} + \sin\theta\ \mathbf{j} - 2r\ \mathbf{k}) \times (-r\sin\theta\ \mathbf{i} + r\cos\theta\ \mathbf{j})\,\mathrm{d}r\mathrm{d}\theta \\
&= \pm\left[-r\sin^2\theta\ \mathbf{j}\times\mathbf{i} + 2r^2\sin\theta\ \mathbf{k}\times\mathbf{i} + r\cos^2\theta\ \mathbf{i}\times\mathbf{j} - 2r^2\cos\theta\ \mathbf{k}\times\mathbf{j}\right]\mathrm{d}r\mathrm{d}\theta \\
&= \pm\left[2r^2\cos\theta\ \mathbf{i} + 2r^2\sin\theta\ \mathbf{j} + r\ \mathbf{k}\right]\mathrm{d}r\mathrm{d}\theta \\
&= \left[2r^2\cos\theta\ \mathbf{i} + 2r^2\sin\theta\ \mathbf{j} + r\ \mathbf{k}\right]\mathrm{d}r\mathrm{d}\theta
\end{aligned}$$

where the positive sign has been chosen to give the outward normal, away from the origin in this case.

3. The surface specification makes no constraints on $\theta$. Expressing $r$ as a function of $z$, we have, $r = \sqrt{4-z}$ for $z > 0$. $r$ must be real. We conclude that,

$$0 \le r \le 2 \qquad \text{and} \qquad 0 \le \theta \le 2\pi.$$

Finally, putting it all together,

$$\begin{aligned}
\Phi &= \iint_S \mathbf{v}\cdot\mathrm{d}S \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \left[r\cos\theta\ \mathbf{i} - (z-4)\ \mathbf{k}\right]\cdot\left[2r^2\cos\theta\ \mathbf{i} + 2r^2\sin\theta\ \mathbf{j} + r\ \mathbf{k}\right]\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \left[r\cos\theta\ \mathbf{i} + r^2\ \mathbf{k}\right]\cdot\left[2r^2\cos\theta\ \mathbf{i} + 2r^2\sin\theta\ \mathbf{j} + r\ \mathbf{k}\right]\mathrm{d}r\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} \left[2r^3\cos^2\theta + r^3\right]\mathrm{d}r\mathrm{d}\theta = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=2} r^3\left[2\cos^2\theta + 1\right]\mathrm{d}r\mathrm{d}\theta \\
&= \left(\int_{\theta=0}^{\theta=2\pi} \cos 2\theta + 2\ \mathrm{d}\theta\right)\left(\int_{r=0}^{r=2} r^3\mathrm{d}r\right) \\
&= \left[\frac{1}{2}\sin 2\theta + 2\theta\right]_{\theta=0}^{\theta=2\pi}\left[\frac{1}{4}r^4\right]_{r=0}^{r=2} = \underline{\underline{16\pi}}
\end{aligned}$$