Module 3A5: Power Generation
Notes on Examples Paper 1

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Abstract

This document outlines the methods used to answer each question on the first Power Generation examples paper. The results are commented on, and extra information is included for interest. This material is intended to complement the more detailed worked solutions available in the official cribs. Statements of physical principles and definitions are highlighted in **bold**, while assumptions are *emphasised*; we are told to treat air as a *perfect gas* throughout.

Q1: Basic gas turbine

Work ratio. First we want an expression for the ratio of compressor work input to the turbine work output. If the turbomachinery is *adiabatic*, then from the **First Law** the work is just the change in enthalpy between the start and end states. Strictly, the relevant change is in stagnation enthalpy, but if the *flow velocities are low* there is no distinction between static and stagnation states. We are not explicitly told that we can make either of these assumptions, but in practice they are accurate enough, and in any case we have none of the information needed to relax them. It is Part I material to determine expressions for the two work flows in terms of isentropic efficiency and temperature changes. Then, an expression for the ratio can be written and simplified to a neat form.

An exercise. Starting from a statement of the **First Law**, deduce non-dimensional ratios which quantify the accuracy of both assumptions made above, that is neglect of kinetic energy and heat loss. These will be of the form,

$$\frac{\text{neglected quantity}}{\text{important quantity}} \ll 1.$$

Efficiencies. To calculate the cycle efficiency, we also need the heat input to the combustor, which is found from applying the **First Law** to it. The Joule cycle is a simpler model of our gas turbine. The standard assumptions are that all processes are reversible, that the working fluid is a perfect gas, and that the turbomachinery is adiabatic. The Joule cycle efficiency depends only on the pressure ratio, and the derivation is Part I material.

Q2: Advanced gas turbine

Air-fuel ratio. In this question, our model of the gas turbine is improved by including the combustion process and accounting for the fuel mass flow. The combustion calculation is very similar to those encountered in the first part of the course. Begin with the **First Law**,

$$(\dot{m}_\mathrm{a} + \dot{m}_\mathrm{f})\, h_\mathrm{p}(T_3) - \dot{m}_\mathrm{a} h_\mathrm{a}(T_2) - \dot{m}_\mathrm{f} h_\mathrm{f}(T_0) = 0, \tag{1}$$

where subscript a denotes air, subscript f denotes fuel and subscript p denotes products. To proceed we need to put this in a form with enthalpy differences, because each of the datum levels for the fuel, air and combustion products are arbitrary. The extra information needed is brought in using an experimental measured value for the standard enthalpy change of reaction, **defined**,

$$\dot{m}_\mathrm{f} \Delta H_0 = (\dot{m}_\mathrm{a} + \dot{m}_\mathrm{f})\, h_\mathrm{p}(T_0) - \dot{m}_\mathrm{a} h_\mathrm{a}(T_0) - \dot{m}_\mathrm{f} h_\mathrm{f}(T_0). \tag{2}$$

Subtracting Eqn. (1) from Eqn. (2), treating the air and combustion products as two different *perfect gases*, and dividing through by $\dot{m}_\mathrm{f}$ yields,

$$\begin{aligned} \dot{m}_\mathrm{f} \Delta H_0 &= \dot{m}_\mathrm{a} \left[h_\mathrm{a}(T_2) - h_\mathrm{a}(T_0)\right] + (\dot{m}_\mathrm{a} + \dot{m}_\mathrm{f}) \left[h_\mathrm{p}(T_3) - h_\mathrm{p}(T_0)\right], \\ \Delta H_0 &= \frac{\dot{m}_\mathrm{a}}{\dot{m}_\mathrm{f}} c_{p,\mathrm{a}} \left[T_2 - T_0\right] + \left(\frac{\dot{m}_\mathrm{a}}{\dot{m}_\mathrm{f}} + 1\right) c_{p,\mathrm{p}} \left[T_3 - T_0\right], \end{aligned}$$

which can be rearranged for the air-fuel ratio $\dot{m}_\mathrm{a}/\dot{m}_\mathrm{f}$. Now we know the air-fuel ratio we do not need to neglect the fuel mass flow in later calculations.

Cycle analysis. Now the combustor is dealt with, we can work out the rest of the states around the cycle and calculate some performance metrics. Fairly straightforward, but we have to be careful with a couple of things. Firstly, the mass flow in the turbine and downstream components is increased by a factor $(1 + \dot{m}_\mathrm{f}/\dot{m}_\mathrm{a})$ relative to the compressor because of the fuel addition. Secondly, the combustion products have different perfect-gas properties to standard air.

Q3: Intercooled cycle

Setting up the problem. The quantity of interest is the total compressor work input. The first step is to get an expression for this dependent variable as a function of the independent variable that we are changing: the first compressor pressure ratio. With the usual method (**First Law**, **isentropic efficiency**, *perfect gas*) the work input for each of the compressors is,

$$-\dot{w}_{x,12} = \frac{c_p T_1}{\eta_\mathrm{c}} \left(r_{12} - 1\right), \tag{3a}$$

$$-\dot{w}_{x,34} = \frac{c_p T_3}{\eta_\mathrm{c}} \left(r_{34} - 1\right), \tag{3b}$$

where r is the isentropic temperature ratio, $r = r_p^{(\gamma-1)/\gamma}$, and the first and second compressors are operating between states $1 \rightarrow 2$ and $3 \rightarrow 4$ respectively. The first compressor outlet temperature, T_2, and hence the second compressor inlet temperature, T_3

depend on the first compressor pressure ratio so T_3 must be eliminated. Using the definitions of **heat exchanger effectiveness** and **isentropic efficiency**,

$$T_3 = T_2 - K\left(T_2 - T_1\right), \qquad \eta_c = \frac{T_{2s} - T_1}{T_2 - T_1} = \frac{r-1}{T_2/T_1 - 1},$$

$$\Rightarrow T_3 = T_1\left[1 - \frac{1}{\eta_c}\left(K-1\right)\left(r_{12}-1\right)\right]. \qquad (4)$$

Then combining Eqns. (3) and (4) the total work input is,

$$-\dot{w}_{x,\mathrm{tot}} = -\dot{w}_{x,12} + -\dot{w}_{x,34},$$

$$-\dot{w}_{x,\mathrm{tot}} = \frac{c_p T_1}{\eta_c}\left[\left(r_{12}-1\right) + \left(r_{34}-1\right) - \frac{1}{\eta_c}\left(K-1\right)\left(r_{12}-1\right)\left(r_{34}-1\right)\right]. \qquad (5)$$

Optimising the pressure ratio. To find the maximum value of Eqn. (5) is a classic stationary points problem–we differentiate and set the derivative to zero. The only complication is that Eqn. (5) contains terms in r_{34} which is a function of r_{12}. We know that the overall pressure ratio is fixed, i.e. $r_{\mathrm{tot}} = r_{12}r_{34} = \mathrm{const.}$, so this could be used to eliminate r_{34}. The neatest way is to deal with the r_{34} terms implicitly, as,

$$\frac{\mathrm{d}}{\mathrm{d}r_{12}}\left(-\dot{w}_{x,\mathrm{tot}}\right) = \frac{c_p T_1}{\eta_c}\left[1 + \frac{\mathrm{d}r_{34}}{\mathrm{d}r_{12}} - \frac{(K-1)}{\eta_c}\left(r_{34}-1\right) - \frac{(K-1)}{\eta_c}\left(r_{12}-1\right)\frac{\mathrm{d}r_{34}}{\mathrm{d}r_{12}}\right],$$

$$\frac{\mathrm{d}}{\mathrm{d}r_{12}}\left(-\dot{w}_{x,\mathrm{tot}}\right) = \frac{c_p T_1}{\eta_c}\left[\left(1 - \frac{(K-1)}{\eta_c}\left(r_{34}-1\right)\right) + \frac{\mathrm{d}r_{34}}{\mathrm{d}r_{12}}\left(1 - \frac{(K-1)}{\eta_c}\left(r_{12}-1\right)\right)\right]$$

Now we find $\mathrm{d}r_{34}/\mathrm{d}r_{12}$,

$$r_{\mathrm{tot}} = r_{12}r_{34} = \mathrm{const.} \;\Rightarrow\; \frac{\mathrm{d}r_{34}}{\mathrm{d}r_{12}} = -\frac{r_{\mathrm{tot}}}{r_{12}^2}.$$

Substituting in and setting to zero for the maximum work yields,

$$\frac{\mathrm{d}}{\mathrm{d}r_{12}}\left(-\dot{w}_{x,\mathrm{tot}}\right) = \frac{c_p T_1}{\eta_c}\left[1 - \frac{(K-1)}{\eta_c}\left(\frac{r_{\mathrm{tot}}}{r_{12}} - 1\right) - \frac{r_{\mathrm{tot}}}{r_{12}^2}\left(1 - \frac{(K-1)}{\eta_c}\left(r_{12}-1\right)\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}r_{12}}\left(-\dot{w}_{x,\mathrm{tot}}\right) = \frac{c_p T_1}{\eta_c}\left[1 - \frac{r_{\mathrm{tot}}}{r_{12}^2} - \frac{(K-1)}{\eta_c}\left(\frac{r_{\mathrm{tot}}}{r_{12}} - 1 - \frac{r_{\mathrm{tot}}}{r_{12}} + \frac{r_{\mathrm{tot}}}{r_{12}^2}\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}r_{12}}\left(-\dot{w}_{x,\mathrm{tot}}\right) = \frac{c_p T_1}{\eta_c}\left(1 - \frac{r_{\mathrm{tot}}}{r_{12}^2}\right)\left(1 + \frac{(K-1)}{\eta_c}\right) = 0. \qquad (6)$$

Equation (6) is satisfied if either of the terms in brackets are zero, that is,

$$r_{12} = \sqrt{r_{\mathrm{tot}}} \quad \text{or} \quad \eta_c = 1 - K.$$

The first of these is the solution we are looking for. The second is not likely to be applicable in practice, as both η_c and K are typically significant fractions of unity, greater than 0.5, say. In the second case, Eqn. (5) reduces to,

$$-\dot{w}_{x,\mathrm{tot}} = \frac{c_p T_1}{\eta_c}\left(r_{\mathrm{tot}} - 1\right),$$

so the total work output is a function only of the total pressure ratio, and not the compression split.

Q4: Effect of turbine entry temperature on efficiency

(a) The cycle efficiency is **defined**,

$$\eta = \frac{W_\mathrm{t} - W_\mathrm{c}}{Q}. \tag{7}$$

If the compressor pressure ratio is maintained, then W_c does not change and η is a function of the two independent variables W_t and Q only. We can then express a small change in cycle efficiency $\delta\eta$ as,

$$\delta\eta = \frac{\partial \eta}{\partial W_\mathrm{t}}\Big|_Q \delta W_\mathrm{t} + \frac{\partial \eta}{\partial Q}\Big|_{W_\mathrm{t}} \delta Q$$

This is a multi-variable, first-order Taylor series expansion as in the Mathematics Data Book. We can evaluate these partial derivatives directly from Eqn. (7), simplify, and divide through by η to get the desired result:

$$\delta\eta = \frac{1}{Q}\delta W_\mathrm{t} - \frac{W_\mathrm{t} - W_\mathrm{c}}{Q^2}\delta Q$$

$$\delta\eta = \frac{\delta W_\mathrm{t}}{Q} - \frac{W_\mathrm{t} - W_\mathrm{c}}{Q}\frac{\delta Q}{Q} \qquad \Rightarrow \qquad \frac{\delta\eta}{\eta} = \frac{\delta W_\mathrm{t}}{\eta Q} - \frac{\delta Q}{Q}$$

(b) Since we have an expression for $\delta\eta$, our task is to put it in terms of δT_3 which will allow us to then divide through and take the limit $\delta T_3 \to 0$ to find $\partial\eta/\partial T_3$. Expressions for δQ and δW_t can either be found by consideration of the T–s diagram, or differentiation of expressions for Q and W_t. The assumption that turbine polytropic efficiency does not change with T_3 allows T_4 to be expressed as a function of T_3. Evaluating the final result gives $\partial\eta/\partial T_3 = 8.6 \times 10^{-5}\,\mathrm{K}^{-1}$, or in more sensible terms we need a temperature increase of order 100 K to get an extra percent on cycle efficiency.

Q5: Recuperated gas turbine

(a) On the vertical axis should be stream temperature, on the horizontal axis fraction of heat transferred. The exhaust is giving up heat to the inlet stream, and vice versa. Pay attention to the slope of each line: think of the **First Law** applied to a differential element of one stream, and what the slope of the graph represents physically.

(b) Applying both the definition of **heat exchanger effectiveness** and the **First Law** over the heat exchanger allows determination of the combustor inlet temperature, $T_{4'}$ in the lecture notes nomenclature. This can then replace T_2 in the expression derived for the air-fuel ratio in Q2. It will be found that the air-fuel ratio increases. This makes sense, because the recuperation has already heated the compressor delivery air somewhat, so less fuel is needed to bring it to the same combustor outlet temperature. We are getting the same work with less external heat addition (if we draw a control volume around the power plant, the recuperator is within it) so the efficiency increases.

(c) The same method from Q2 can be used to calculate the exhaust temperature and hence exergy flow rate. These are both reduced, quantitatively reflecting the benefit of recuperation.

Q6: Effect of coolant flow rate on efficiency

(a) With the usual assumptions, we have the following expressions for the turbine work,

$$\text{without cooling:} \qquad W_\mathrm{t} = c_p\,(T_3 - T_4)\,, \tag{8a}$$

$$\text{with cooling:} \qquad W_\mathrm{t} = (1-\delta m)\,c_p\,(T_3 - T_\mathrm{m}) + c_p\,(T_\mathrm{mix} - T_\mathrm{4m})\,, \tag{8b}$$

using the state nomenclature from the lecture notes. There are two unknowns in these equations: T_mix and T_4m. The former can be found by applying the **First Law** to the *constant-pressure* mixing process. The latter can be found assuming the *polytropic efficiency is not a function of mass flow*, which fixes the temperature ratio across the second part of the expansion. The change in turbine work can then be found by subtracting Eqns. (8b) and (8a).

(b) The heat input is a function of mass flow and combustor inlet and outlet temperatures. T_2 and T_3 are fixed so the only contribution to the change in heat input δQ is due to the reduction in mass flow of δm. Substituting expressions for δW_t and δQ into the first result of Q4 yields the answer.

(c) For a small change in efficiency due to both changes in T_3 and δm,

$$\delta\eta = \left.\frac{\partial\eta}{\partial T_3}\right|_m \delta T_3 + \left.\frac{\partial\eta}{\partial m}\right|_{T_3} \delta m.$$

The partial derivatives have been worked out previously, in Q4(b) and Q6(b) respectively. Substituting these in and dividing through by δT_3 yields the required result when $\delta T_3 \to 0$.

(d) What the question is asking for is the range of values for $\mathrm{d}m/\mathrm{d}T_3$ which satisfy the inequality $\partial\eta/\partial T_3|_r > 0$ at the values of temperature and cycle efficiency associated with a particular gas turbine. This comes out at $\mathrm{d}m/\mathrm{d}T_3 \geq 1365\,\mathrm{K}$, i.e. to bother with cooling at all we should be able to increase T_3 by at least 13 K for every 1% of cooling air used.

(e) The other advantage of increasing T_3 is that the specific work output increases. This means that less mass flow is required for the same power, and hence the flow area can be reduced, and the machine can be reduced in overall size. As cost (very approximately) scales with the volume of the machine, this is an important consideration.